TREK Tracking by the Kalman Filter

Tongtong Cao
Hampton University

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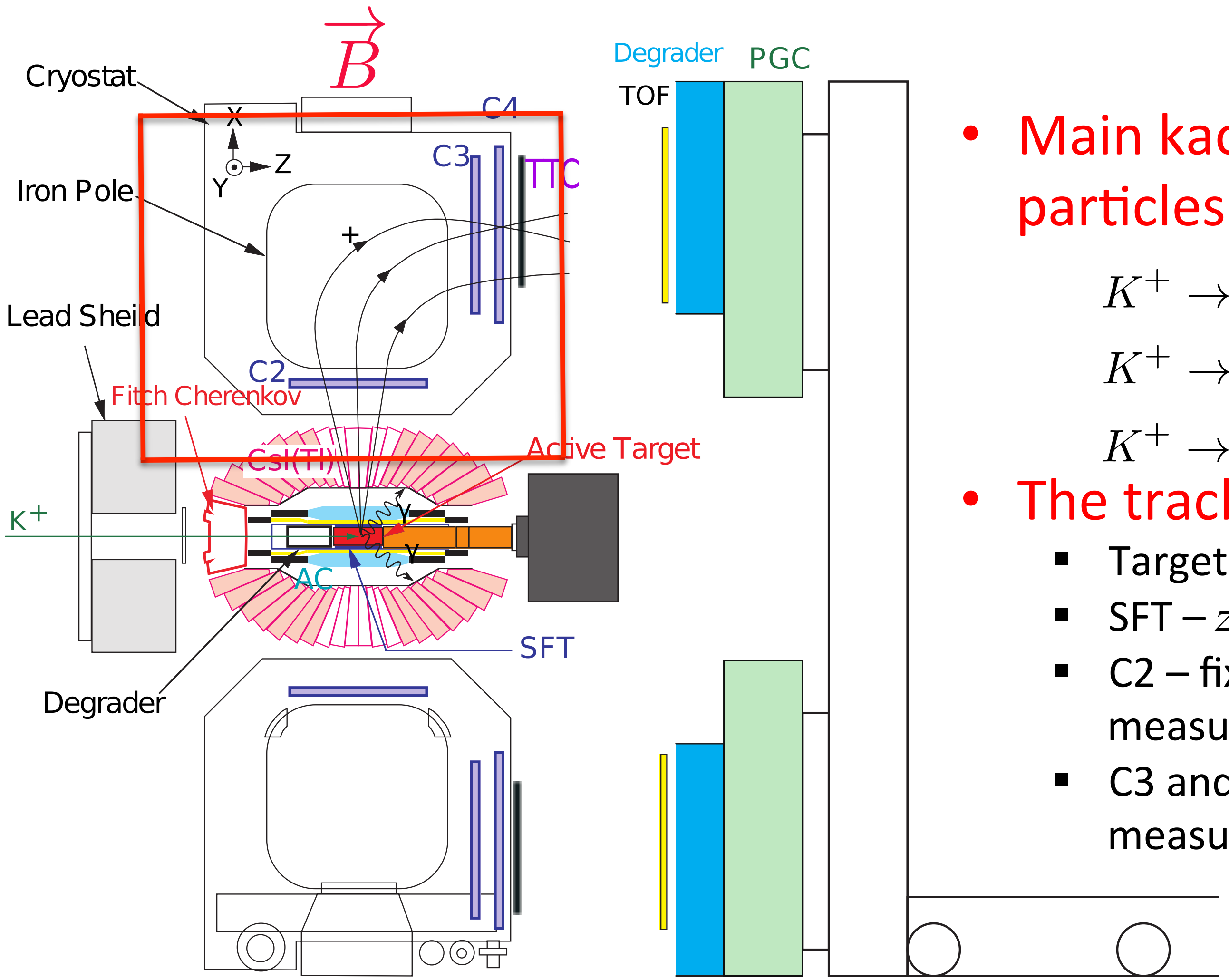
Outline

- Particles’ trajectories of the TREK experiment
- Introductions to Kalman filter and smoother
- TREK tracking with application of Kalman filter and smoother
- TREK tracking program package
- Test and preliminary results
- Summary

The Tracking System of the TREK Experiment



- Main kaon decays (charged particles are detected):

$$K^+ \to \mu^+\nu \quad (63.55\%)$$

$$K^+ \to e^+\nu \quad (0.001582\%)$$

$$K^+ \to \pi^+\pi^0 \quad (20.67\%)$$

- The tracking system:
  - Target – x and y measurement
  - SFT – z measurement
  - C2 – fixed x plane; y and z measurement
  - C3 and C4 – fixed z plane; x and y measurement

Physics on the Charged Particles' Trajectories

- Lorentz force in the B area: momentum direction is changed
- Ionization and Bremsstrahlung radiation: energy is lost and momentum magnitude becomes smaller and smaller
- Multi-scattering: momentum direction is slightly changed at random

Idea of the Kalman Filter

The Kalman filter proceeds progressively from one measurement to the next and improves the knowledge about the particle trajectory by updating the track parameters with each new measurement.

- The state-space system from step *k-1* to *k*: $\tilde{x}_k = F_{k-1}x_{k-1} + w_{k-1}$, $m_k = H_k\tilde{x}_k + \nu_k$
  - F_{k-1} is propagation matrix with consideration of Lorentz force;
  - w_{k-1} denotes random process noise with consideration of multi-scattering, which is regarded as Gaussian distribution with zero mean
  - H_k is projector matrix
  - v_k denotes measurement noise
- Estimated state at step *k* by propagation :$x_k^{k-1} = F_{k-1}x_{k-1}$
- Estimated covariance matrix at step *k* by propogation :

$$C_k^{k-1} = E[(\tilde{x}_k - x_k^{k-1})(\tilde{x}_k - x_k^{k-1})^T] = F_{k-1}C_{k-1}F_{k-1}^T + Q_{k-1} \quad (Q_{k-1} \equiv w_{k-1}w_{k-1}^T)$$

- The updated state by the filter: $x_k = x_k^{k-1} + K_k(m_k - H_k x_k^{k-1})$
  - $m_k - H_k x_k^{k-1}$ is called the innovation term, which is residual of the real measurement and the expected measurment
  - K_k is called the Kalman gain matrix, which represents a measurement by how much the innovation improves the expected state.
- The sum of the variances of the estimation errors:

$$J_k = TrC_k \text{, where } C_k = E[(\tilde{x}_k - x_k)(\tilde{x}_k - x_k)^T]$$

- To minimize J_k, $\partial J_k / \partial K_k = 0$, then

$$K_k = C_k^{k-1}H_k^T(H_k C_k^{k-1}H_k^T + V_k)^{-1}(V_k \equiv \nu_k\nu_k^T),\ C_k = (I - K_k H_k)C_k^{k-1}$$

Extended Kalman Filter

Prediction equations:

$$x_k^{k-1} = f_{k-1}(x_{k-1})$$

$$C_k^{k-1} = F_{k-1} C_{k-1} F_{k-1}^T + Q_{k-1}$$

$$F_{k-1} = \frac{\partial f_{k-1}}{\partial x}\Big|_{x_{k-1}}$$

Filter equations:

$$x_k = x_k^{k-1} + K_k(m_k - h_k(x_k^{k-1}))$$

$$K_k = C_k^{k-1} H_k^T (H_k C_k^{k-1} H_k^T + V_k)^{-1}$$

$$C_k = (I - K_k H_k) C_k^{k-1}$$

$$H_k = \frac{\partial h_k}{\partial x}\Big|_{x_k}$$

x: state vector
F: propagation matrix
C: covariance matrix of state
Q: covariance matrix of process noise

m: measurement vector
V: covariance matrix of measurement noise
H: projector matrix
K: Kalman gain matrix

χ^2

In case of the Kalman filter, there are two residuals to consider: The first one is the residual between the expected state vector x_k^{k-1} after propagation from measurement *k-1* to *k* and the state vector x_k after filtering. The second one is the residual of the actual measurement m_k and the measurement expected from the filtered estimate $h_k(x_k)$.

$$\chi_k^2 = \chi_{k-1}^2 + (x_k - x_{k-1})^T (C_k^{k-1})^{-1} (x_k - x_{k-1}) + \\ (m_k - h_k(x_k))^T (V_k)^{-1} (m_k - h_k(x_k))$$

Smoother

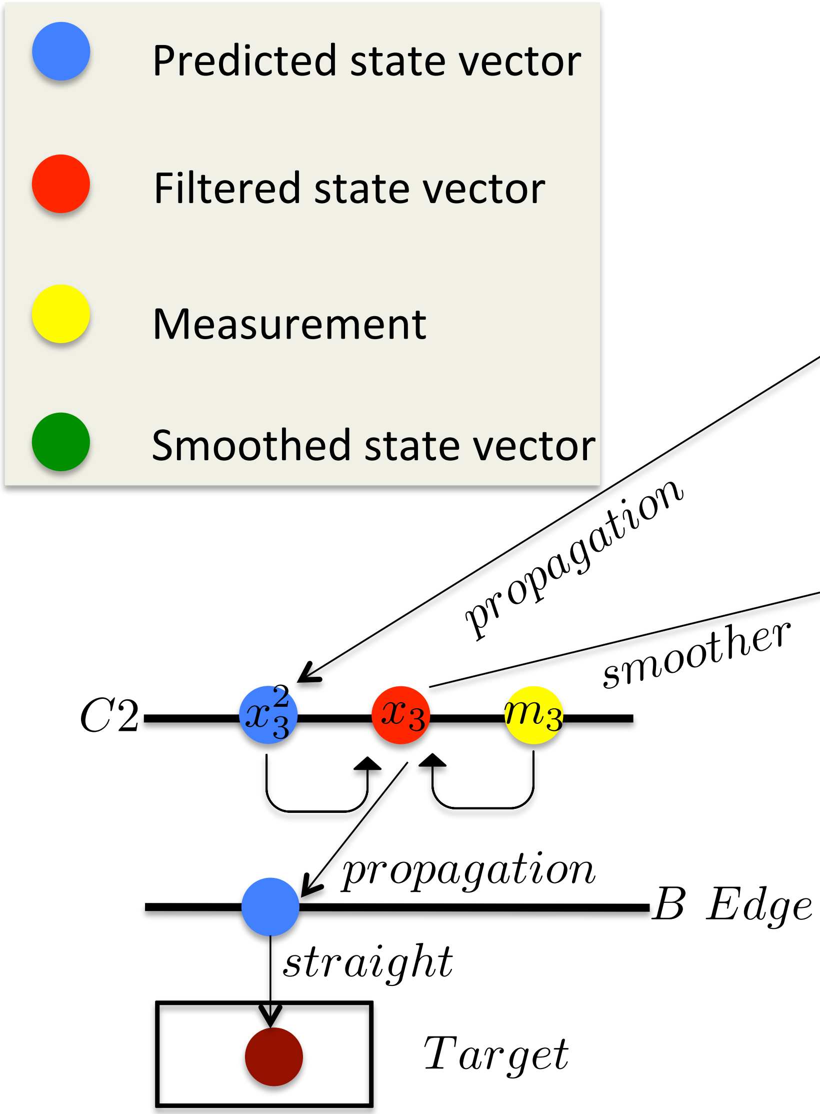
- Due to the recursive nature of the Kalman filter approach the computed state vector x_k is based on the k measurements collected so far. It is unaffected by subsequent estimates. A smoother allows to further improve this estimate using information from any subsequent measurements as well.
- The smoothing is also a recursive procedure which proceeds step by step in the direction opposite to that of the filter with the smoother equations:

$$x_k^n = x_k + A_k(x_{k+1}^n - x_{k+1}^k)$$

$$A_k = C_k F_k^T (C_{k+1}^k)^{-1} \qquad k \in \{1, \cdots, n-1\}$$

$$C_k^n = C_k + A_k(C_{k+1}^n - C_{k+1}^k)A_k^T$$

Scheme for Current TREK Tracking



- Kalman filter and smoother:
  - The prediction makes an estimate of the future state of the system.
  - The filter estimates the system state in the present.
  - The smoother allows to estimate the state of the system in the past.
- Extend to target: The filtered state vector at C2 propagates to the lower edge of the magnetic field, and then construct a straight line to determine the vertex with the no B area.

State Vector for TREK and Propagation Equations

State vector: $(x, y, z, \theta, \varphi, q/p)^T$

Reference: s (path length)

Lorentz force law: $\frac{d\vec{n}}{ds} = \kappa \frac{q}{p} \vec{n} \times \vec{B}$

$$\kappa = 0.299792458 \frac{MeV}{c \cdot mm \cdot T}$$

$$\begin{cases} n_x = \sin\theta \cos\varphi \\ n_y = \sin\theta \sin\varphi \\ n_z = \cos\theta \end{cases}$$

Propagation function:

$$g = \begin{cases} dx/ds = n_x \\ dy/ds = n_y \\ dz/ds = n_z \\ dn_x/ds = w_z n_y - w_y n_z \\ dn_y/ds = w_x n_z - w_z n_x \\ dn_z/ds = w_y n_x - w_x n_y \\ d(q/p)/ds = 0 \end{cases}$$

$$\begin{cases} dn_x/ds = \frac{\partial n_x}{\partial \theta}\frac{\partial \theta}{\partial s} + \frac{\partial n_x}{\partial \varphi}\frac{\partial \varphi}{\partial s} \\ dn_y/ds = \frac{\partial n_y}{\partial \theta}\frac{\partial \theta}{\partial s} + \frac{\partial n_y}{\partial \varphi}\frac{\partial \varphi}{\partial s} \\ dn_z/ds = \frac{dn_z}{d\theta}\frac{d\theta}{ds} \end{cases}$$

$$\xrightarrow{\text{Any of two equations are equivalent}}$$

$$g = \begin{cases} dx/ds = \sin\theta\cos\varphi \\ dy/ds = \sin\theta\sin\varphi \\ dz/ds = \cos\theta \\ d\theta/ds = w_x \sin\varphi - w_y \cos\varphi \\ d\varphi/ds = \cot\theta(w_x\cos\varphi + w_y\sin\varphi) - w_z \\ d(q/p)/ds = 0 \end{cases}$$

$$\begin{cases} w_x = \kappa \frac{q}{p} B_x \\ w_y = \kappa \frac{q}{p} B_y \\ w_z = \kappa \frac{q}{p} B_z \end{cases}$$

Propagation: Adaptive Forth-Order Runge-Kutta Method

To avoid the problems of the fixed step length methods, and to adjust the accuracy in a predictable way, an adaptive Runge-Kutta method is needed.

$$
\begin{aligned}
k_1^{(n)} &= f\left(x_n, t_n\right) \\
k_2^{(n)} &= f\left(x_n + \frac{1}{2} h_n k_1^{(n)}, t_n + \frac{1}{2} h_n\right) \\
k_3^{(n)} &= f\left(x_n + \frac{1}{2} h_n k_2^{(n)}, t_n + \frac{1}{2} h_n\right) \\
k_4^{(n)} &= f\left(x_n + h_n k_3^{(n)}, t_n + h_n\right) \\
x_{n+1} &= x_n + \frac{1}{6} h_n \cdot \left(k_1^{(n)} + 2k_2^{(n)} + 2k_3^{(n)} + k_4^{(n)}\right) + \mathcal{O}\left(h_n^5\right)
\end{aligned}
$$

Error estimation: $\epsilon^{(n)} = h_n^2 (k_1^{(n)} - k_2^{(n)} - k_3^{(n)} + k_4^{(n)})$

- The step size h is automatically adjustable in the programming to meet the requirement of the accuracy.
- The propagation routine is designed to be bidirectional.

Propagation for x and F

Let $h = s_e - s_0$ be the step size. The forth-order Runge-Kutta method numerically computes the change in the track state $\Delta x = x(s_e) - x(s_0)$ from $x(s_0)$ to $x(s_e)$ by evaluating the propagation function g at four points with different s-coordinates

$$s_i = \{s_0, s_0 + \frac{1}{2}h, s_0 + \frac{1}{2}h, s_e\}$$

x propagation

$$x(s_e) = x(s_0) + \frac{1}{6}\Delta x_1 + \frac{1}{3}\Delta x_2 + \frac{1}{3}\Delta x_3 + \frac{1}{6}\Delta x_4$$

$$\begin{cases} \Delta x_1 = hg(x(s_0), s_0) \\ \Delta x_2 = hg(x(s_0) + \frac{1}{2}\Delta x_1, s_0 + \frac{1}{2}h) \\ \Delta x_3 = hg(x(s_0) + \frac{1}{2}\Delta x_2, s_0 + \frac{1}{2}h) \\ \Delta x_4 = hg(x(s_0) + \Delta x_3, s_0 + h) \end{cases}$$

Propagation matrix F

$$F = I + \frac{1}{6}F_1 + \frac{1}{3}F_2 + \frac{1}{3}F_3 + \frac{1}{6}F_4$$

$$F_i = h\frac{dg(x_i, s_i)}{dx_i}(I + F_{i-1}\frac{s_i - s_0}{h})$$

$$\frac{dg(x_i, s_i)}{dx_i} = \begin{bmatrix} 0 & 0 & 0 & \cos\theta\cos\varphi & -\sin\theta\sin\varphi & 0 \\ 0 & 0 & 0 & \cos\theta\sin\varphi & \sin\theta\cos\varphi & 0 \\ 0 & 0 & 0 & -\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & w_x\cos\varphi + w_y\sin\varphi & \frac{(w_x\sin\varphi - w_y\cos\varphi)}{(q/p)} \\ 0 & 0 & 0 & -\csc^2\theta(w_x\cos\varphi + w_y\sin\varphi) & \cot\theta(-w_x\sin\varphi + w_y\cos\varphi) & \frac{\cot\theta(w_x\cos\varphi + w_y\sin\varphi) - w_z}{(q/p)} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Energy Loss, Multi-Scattering and Q

Energy loss and multi-scattering is calculated and momentum magnitude is updated for each Runge-Kutta step along the trajectory.

Energy loss due to ionization

For u^+ and π^+: Bethe-Bloch formula

$$-\frac{dE}{ds} = Kz^2 \frac{Z}{A}\frac{1}{\beta^2}(\frac{1}{2}ln(\frac{2m_e c^2 \beta^2 \gamma^2 T_{max}}{I^2}) - \beta^2)$$

$$K = 4\pi N_A r_e^2 m_e C^2 = 0.307075 \frac{MeV \cdot cm^2}{g}$$

For e^+: $$-\frac{dE}{ds} = \frac{1}{2}K\frac{Z}{A}(2ln\frac{2m_e c^2}{I} + 4ln\gamma - 2)$$

Valid range for both formulas: $\beta\gamma \in [0.1, 1000]$, within momentum range of TREK

Energy loss due to radiation is negligibly small

Multi-Scattering

$$\theta_{ms} = \frac{13.6}{\beta cp} z \sqrt{\frac{s}{X_0}}(1 + 0.038 ln\frac{s}{X_0})$$

Q

$$Q(\theta,\theta) = \theta_{ms}^2$$

$$Q(\varphi,\varphi) = \frac{1}{\sin^2\theta}\theta_{ms}^2$$

Other elements are 0

F and Q between Two Measurements

Suppose that there are n Runge-Kutta steps (step = $1, 2, \ldots, n$) between two measurements:

- $F = F_n F_{n-1} \cdots F_2 F_1$
- $Q = Q_1 + Q_2 + \cdots + Q_n$

m, V and H

Measurement vector:

$$m = (x, y, z)^T$$

Measurement noise matrix:

$$V = \begin{bmatrix} \sigma_x^2 & 0 & 0 \\ 0 & \sigma_y^2 & 0 \\ 0 & 0 & \sigma_z^2 \end{bmatrix}$$

Projector matrix:

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Initial State Vector and Covariance Matrix

The optimized setup of initial state and covariance is the key to make the tracking successful since the number of measurements is very limited.

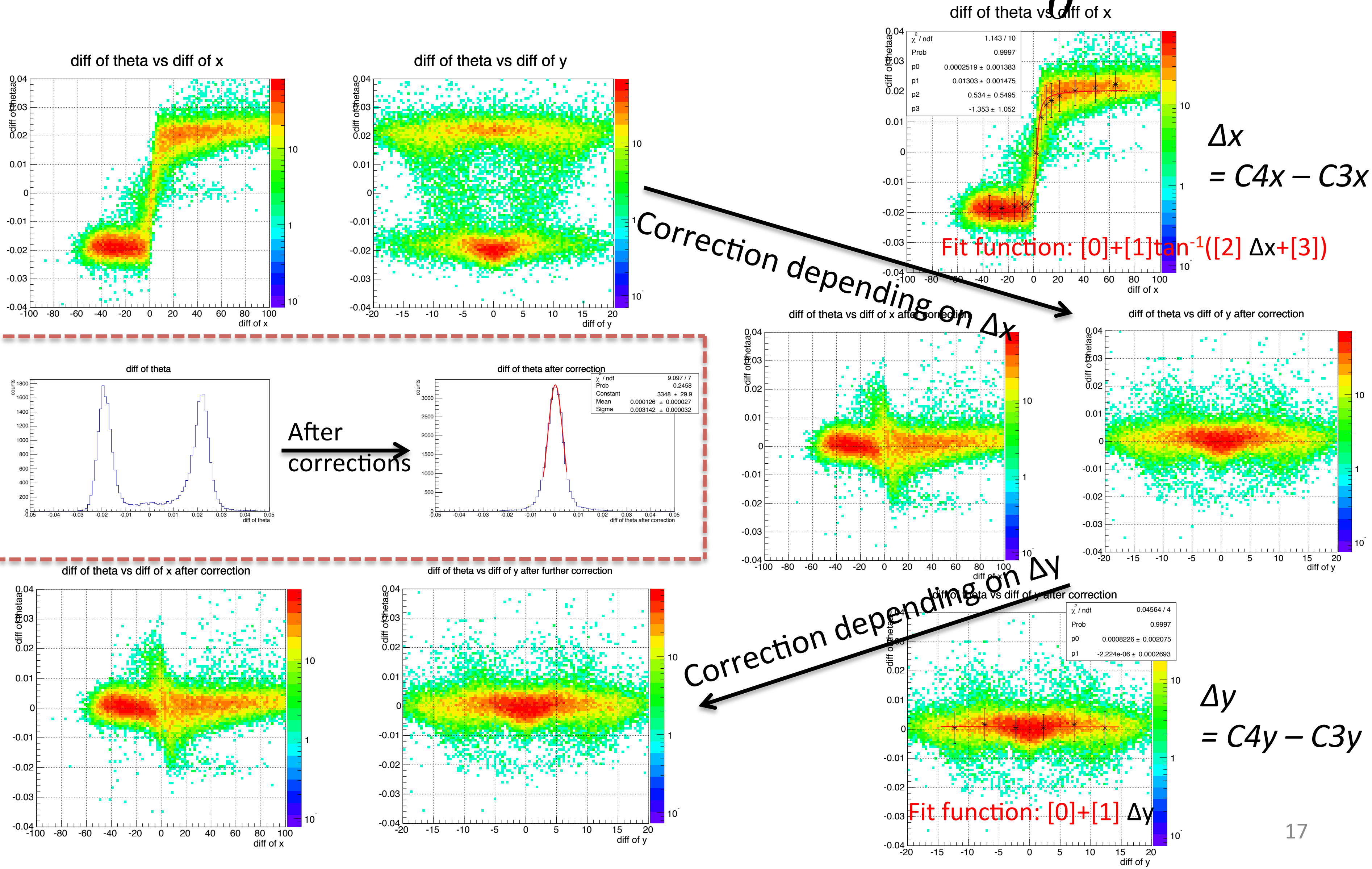
- Initial state vector: $(x_0, y_0, z_0, \theta_0, \varphi_0, q/p_0)$
- Initial covariance matrix:

$$\begin{bmatrix} Var(x_0) & 0 & 0 & 0 & 0 & 0 \\ 0 & Var(y_0) & 0 & 0 & 0 & 0 \\ 0 & 0 & Var(z_0) & 0 & 0 & 0 \\ 0 & 0 & 0 & Var(\theta_0) & 0 & 0 \\ 0 & 0 & 0 & 0 & Var(\varphi_0) & 0 \\ 0 & 0 & 0 & 0 & 0 & Var(q/p_0) \end{bmatrix}$$

- x_0, y_0, z_0, $Var(x_0)$, $Var(y_0)$ and $Var(z_0)$ are the same as measurements and square of uncertainties at C4.
- θ_0 and φ_0 are determined by the straight line between measurements of C4 and C3 since the magnetic field is weak between them.

Simulated Data for Kμ2

Correction of Initial θ_0



Initial Variances of θ_0 and φ_0

Initial variances of θ_0 and φ_0 are from two aspects:

1. The first aspect is the difference of angle values between estimation and true, which is estimated by the simulated data
2. The second aspect is caused by uncertainties of C4 and C3 measurements, which is directly given by analytical solutions.

Set C4 and C3 measurements as (*x1*, *y1*, *z1*) and (*x2*, *y2*, *z2*) and their uncertainties as $(\sigma_{x1}, \sigma_{y1}, \sigma_{z1})$ and $(\sigma_{x2}, \sigma_{y2}, \sigma_{z2})$

Then, $\Delta x = x1 - x2$, $\Delta y = y1 - y2$, $\Delta z = z1 - z2$,

and
$$\sigma_{\Delta x} = \sqrt{\sigma_{x1}^2 + \sigma_{x2}^2},\ \sigma_{\Delta y} = \sqrt{\sigma_{y1}^2 + \sigma_{y2}^2},\ \sigma_{\Delta z} = \sqrt{\sigma_{z1}^2 + \sigma_{z2}^2}$$

For θ,
$$\cos\theta = \frac{\Delta z}{\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}}$$
$$var(\theta) = (\frac{\partial\theta}{\partial\Delta x})^2\sigma_{\Delta x}^2 + (\frac{\partial\theta}{\partial\Delta y})^2\sigma_{\Delta y}^2 + (\frac{\partial\theta}{\partial\Delta z})^2\sigma_{\Delta z}^2$$

For φ,
$$\tan\varphi = \frac{\Delta y}{\Delta x}$$
$$var(\varphi) = (\frac{\partial\varphi}{\partial\Delta x})^2\sigma_{\Delta x}^2 + (\frac{\partial\varphi}{\partial\Delta y})^2\sigma_{\Delta y}^2$$

Seed, q/p_0 and $Var(q/p_0)$

- Each seed includes 3 MWPC measurements.

- To determine initial momentum magnitude of a seed, a routine is built to choose the best initial momentum with the minimized χ^2. The idea: apply momentum magnitude trials and extract χ^2 for each trial, then decrease the range of candidate momentum magnitude by comparison of χ^2.

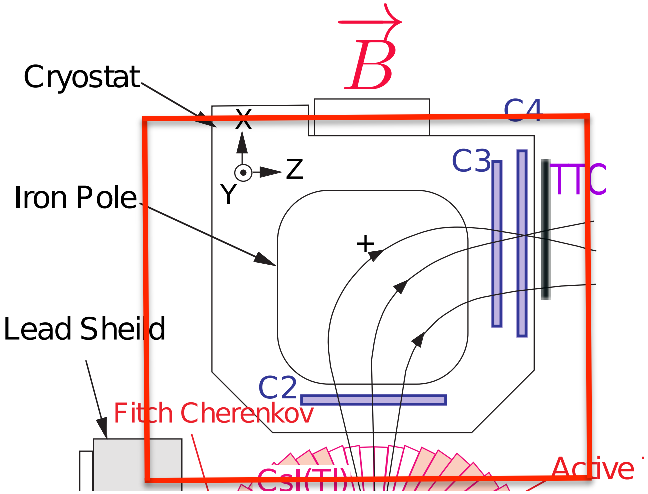
- The uncertainty of the momentum magnitude candidate with the range of 92.5 to 347.5 MeV/c is 0.25 MeV/c, so $Var(q/p_0)$ is $(0.25)^2 q^2/p_0^4$.

Fast tracker

To decrease number of momentum magnitude trials, a fast tracker is applied:

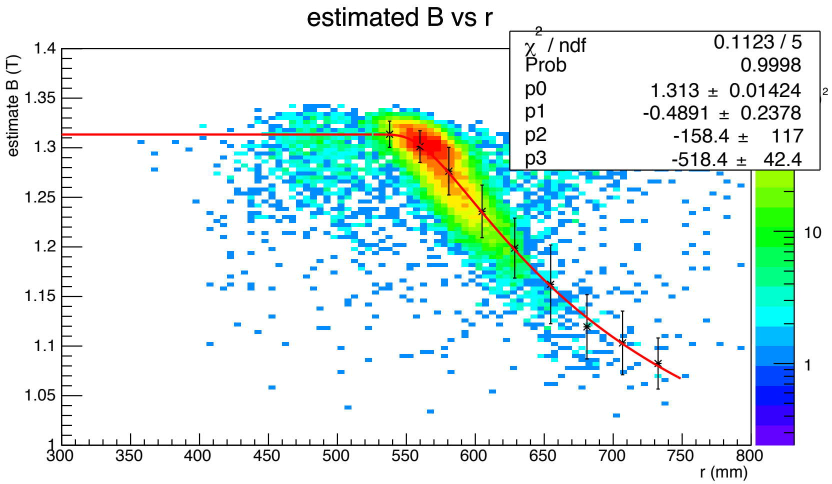
1. In the *x-z* plane, the trajectory between C2 and C3 is regarded as a circular arc, while the circular center is located at the straight line which is vertical to the C4-C3 straight line

2. Then $p = kBrc/\sqrt{n_x^2 + n_z^2}$,
   where *r* is radius of the circle,
   and $\kappa = 0.299792458 \dfrac{MeV}{c \cdot mm \cdot T}$



Application of Fast Tracker

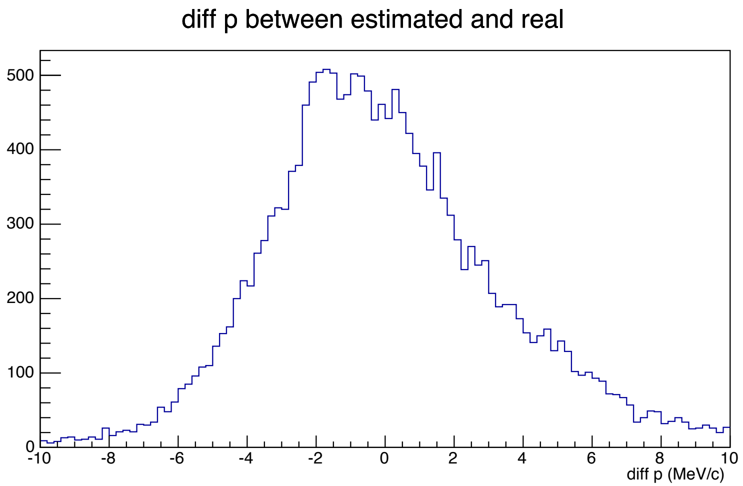
$\vec{B}$ is inhomogeneous



B determination function:

$$B = f(r) = [0] + [1]e^{\frac{[2]}{r+[3]}} \quad (r > 525\ mm)$$

$$B = f(525) \qquad (r <= 525\ mm)$$

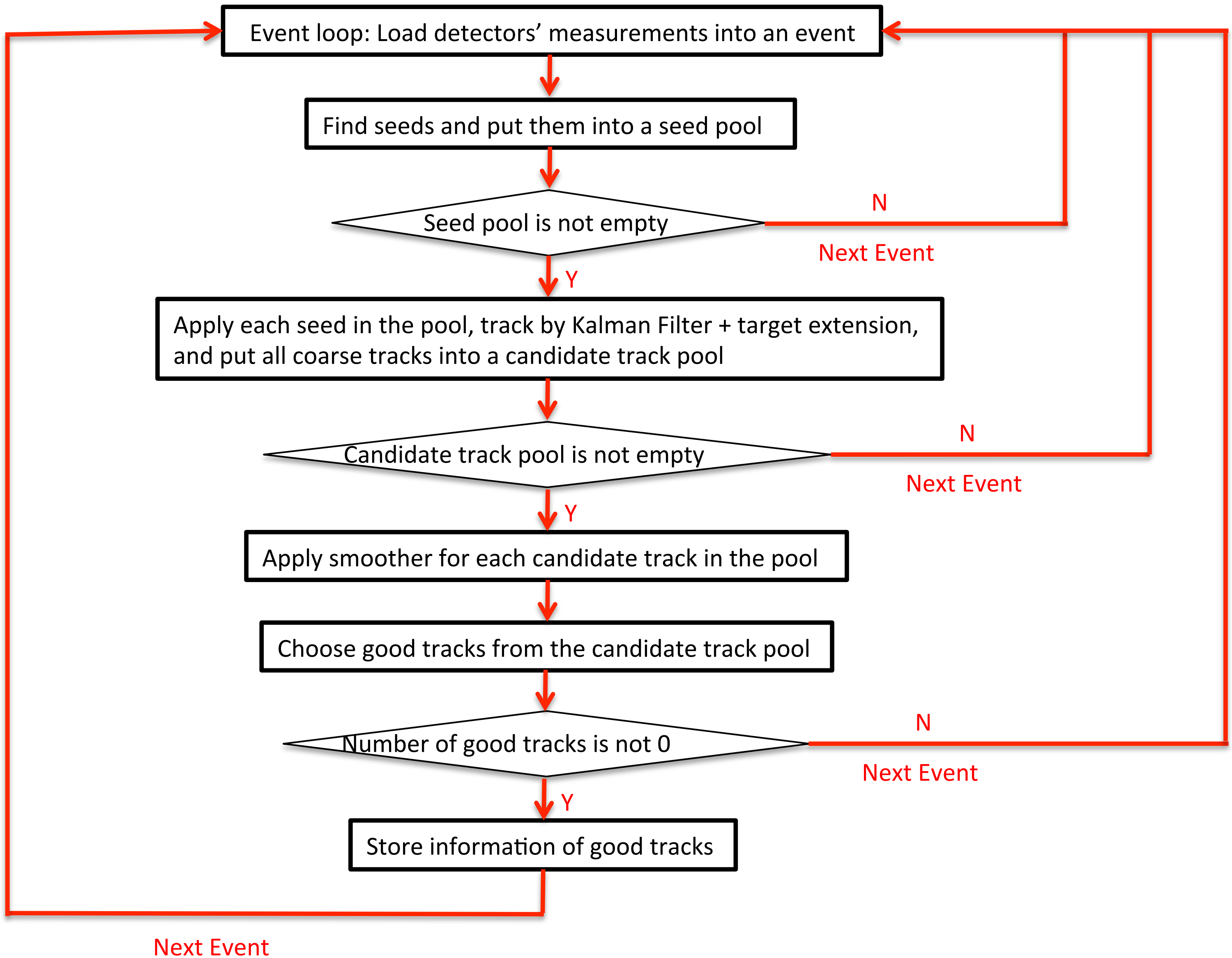


- If the best momentum candidate is judged to appear between estimated momentum +/- 8 MeV/c, an improved routine with less trials is applied to extract the best momentum candidate.
- Otherwise, the previous general routine with more trials is applied.

Why Develop A Tracking Package from Scratch?

- Public Kalman filter packages, like GENFIT, serves for two cases,
  1. Particle's trajectory is helix
  2. Particle's trajectory has small deflection angle which is less than 90^0
- For most events of the TREK experiment, the deflection angle is large than 90^0.
- For the public packages, state vectors are represented by 5 parameters with a specified reference, including 2 parameters for position, 2 parameters for momentum direction, and 1 parameter for momentum magnitude. For the TREK experiments, the state vector requires 3 parameters for position, so there are totally 6 parameters with the path length as reference.
- The third position parameter is regarded as the location of the detector plane with a very small uncertainty. The MWPC hit position is an average of several tens to several hundreds of ion-electron pairs generated positions in a particle's path within the detector's effective area. The average value is around the detector plane and has an uncertainty.

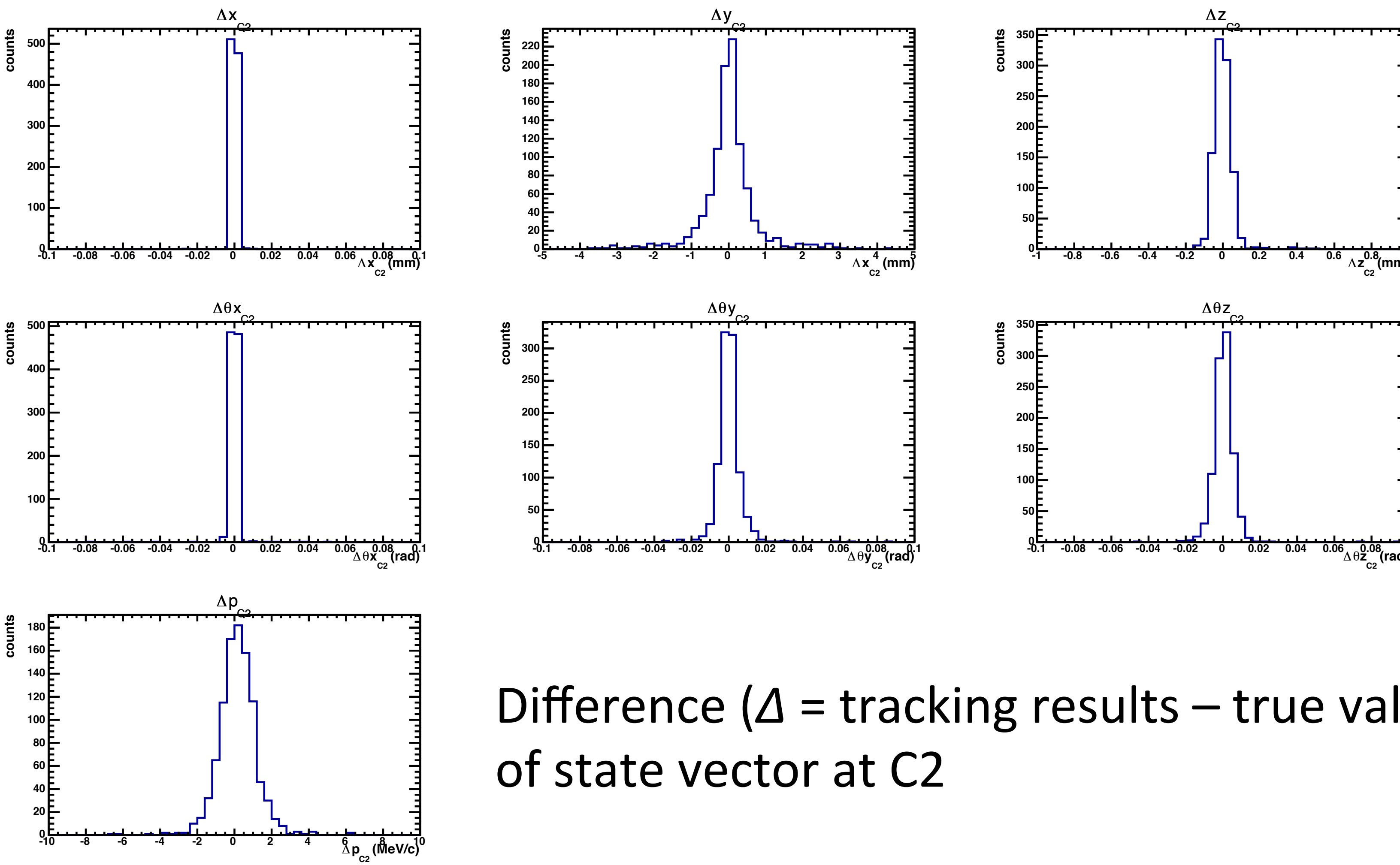
Scheme of the Tracking Package



Overview of Programming

- Environment: C++ and ROOT
- Build tool: Make
- Main function: trekTracker.cxx
- Constant value setup: utility.h
- Classes:
  - Particles' properties: particle
  - Plane and hit: plane and planeHit
  - Detectors' properties: mwpcs, tof1, tof2, sft, target, etc
  - Field map and propagation: fieldMap and fieldStepper
  - Specified matrix definition for TREK tracking: matrix
  - Tracking by Kalman filter and smoother: trackSystem, trackSite and trackState
  - Tracking engine: trackFinder
  - Track information storage: track

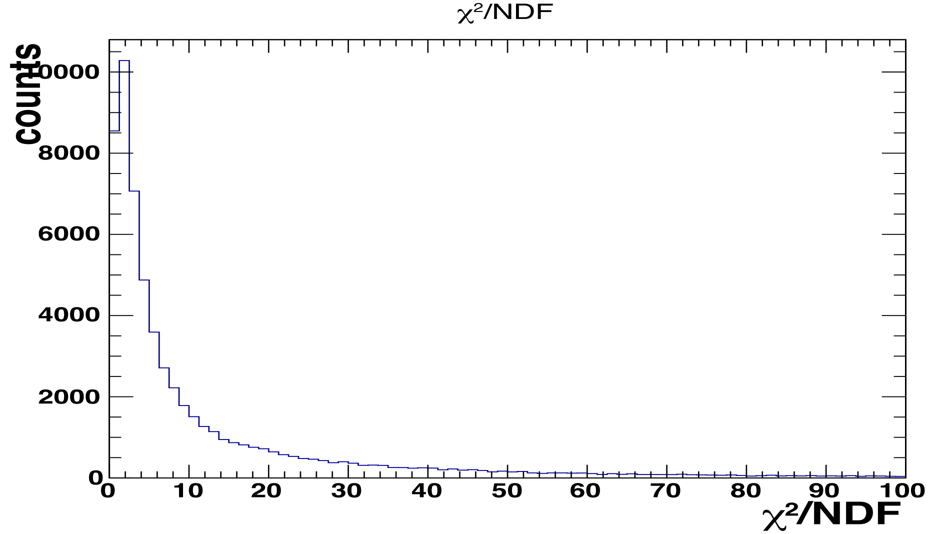
Application for Simulated Data

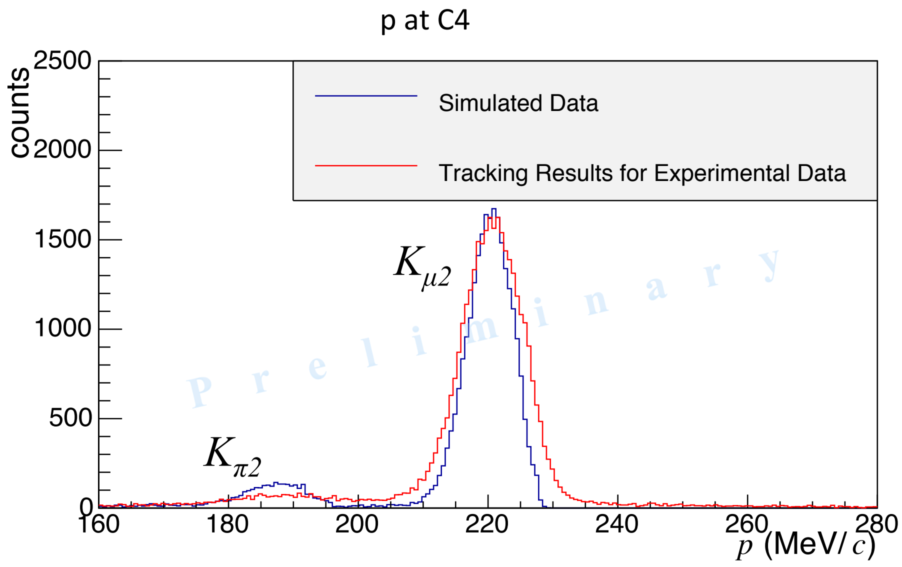


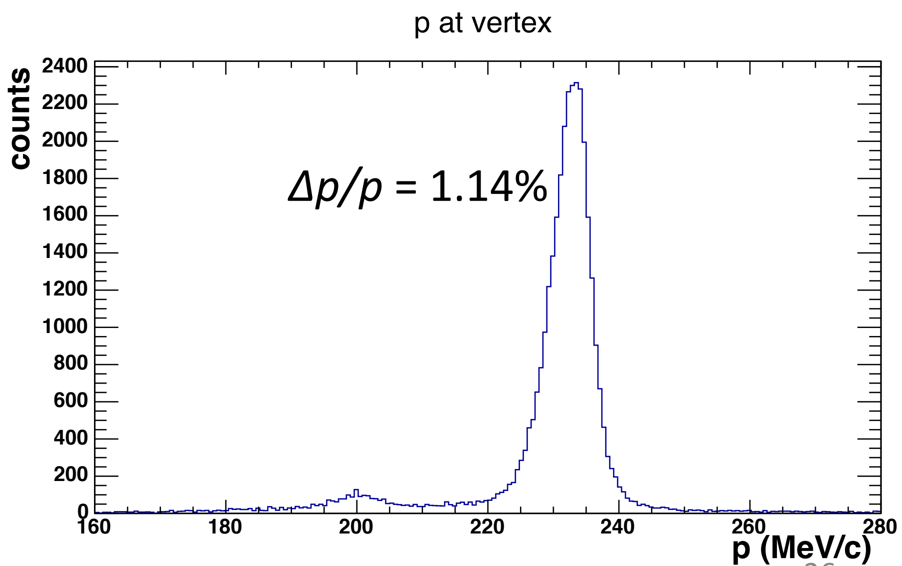
Difference (*Δ* = tracking results – true values) of state vector at C2

Application for Experimental Data

Tracking efficiency = 97.6%







Summary and Future

- The TREK tracking package with application of the Kalman filter is developed and works well.
- To further improve the momentum resolution
  - The package will be updated to directly apply the target and SFT measurements into the Kalman filter.
  - The setup for geometry and material of the detector system will be polished.